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FISCHER DECOMPOSITION FOR DIFFERENCE DIRAC OPERATORS

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Abstract. *We establish the basis of a discrete function theory starting with a Fischer decomposition for difference Dirac operators. Discrete versions of homogeneous polynomials, Euler and Gamma operators are obtained. As a consequence we obtain a Fischer decomposition for the discrete Laplacian.*

1 INTRODUCTION

Clifford analysis is a powerful tool to solve all kinds of problems related with vector field analysis.
A comprehensive description of Clifford function theory was given by F. Brackx, R. Delanghe and F. Sommen in [1] and later by R. Delanghe, F. Sommen and V. Souček in [2].

In [5, 6], K. Gürlebeck and W. Sprößig proposed strategies to solve boundary value problems based on the study of existence, uniqueness, representation, and regularity of solutions with the help of an operator calculus. In the same books, the authors introduce also the basic ideas to develop a discrete counterpart to the continuous treatment of boundary value problems with the introduction of a discrete operator calculus in order to find a well-adapted numerical approach. An explicit discrete version of the Borel-Pompeiu formula was presented for dimension $n = 3$.

This was further developed in [7, 8], where K. Gürlebeck and A. Hommel developed finite difference potential methods in lattice domains based on the concept of discrete fundamental solutions for the difference Dirac operator which generalizes the work developed by Ryabenkij in [10]. A numerical application of this theory was presented recently by N. Faustino, K. Gürlebeck, A. Hommel, and U. Kähler in [3] for the incompressible stationary Navier-Stokes equations. In this paper, the authors proposed a scheme which solves efficiently problems in unbounded domains and show the convergence of the numerical scheme for functions with Hölder regularity which is a better gain compared with the convergence results for classical difference schemes.

Moreover, while all these papers claim to be based on discrete function theoretical approaches, from the concepts of the theory of monogenic functions only the Borel-Pompeiu formula and with it Cauchys integral formula were obtained. There is no “real” development of a discrete monogenic function theory up to now.

This paper is supposed to be a step in this direction. To this end discrete versions of a Fischer decomposition, Euler and Gamma operators are obtained. For the sake of simplicity we consider in the first part only Dirac operators which contain only forward or backward finite differences. Of course, these Dirac operators do not factorize the classic discrete Laplacian. Therefore, we will consider in the last chapter a different definition of a difference Dirac operator in the quaternionic case (c.f. [7]) which do factorizes the discrete Laplacian.

Let us emphasize in the end a major obstacle in the discrete case. While in the continuous case the are only one partial derivative for each coordinate x_j we have two finite differences in the discrete case. Therefore, we will have not only one Euler or Gamma operator as in the continuous case, but several. Each one will turn out to be connected to one particular Dirac operator.

2 PRELIMINARIES

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be an orthonormal basis of R^n. The Clifford algebra $C\ell_{0,n}$ is the free algebra over R^n generated modulo the relation

$$x^2 = -|x|^2\mathbf{e}_0,$$

where $\mathbf{e}_0$ is the identity of $C\ell_{0,n}$. For the algebra $C\ell_{0,n}$ we have the anti-commutation relationship

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}\mathbf{e}_0,$$

where δ_{ij} is the Kronecker symbol. In the following we will identify the Euclidean space R^n with $\bigwedge^1 C\ell_{0,n}$, the space of all vectors of $C\ell_{0,n}$. This means that each element x of R^n may be represented by

$$x = \sum_{i=1}^{n} x_i \mathbf{e}_i.$$

From an analysis viewpoint one extremely crucial property of the algebra $C\ell_{0,n}$ is that each non-zero vector $x \in R^n$ has a multiplicative inverse given by $\frac{-x}{|x|^2}$. Up to a sign this inverse corresponds to the Kelvin inverse of a vector in Euclidean space. Moreover, given a general Clifford number $a = \sum_A \mathbf{e}_A a_A, A \subset \{1, \ldots, n\}$ we denote by $\mathrm{Sc}\, a = a_\emptyset$ the scalar part and by $\mathrm{Vec}\, a = \mathbf{e}_1 a_1 + \ldots + \mathbf{e}_n a_n$ the vector part.

We now introduce the Dirac operator $D = \sum_{i=1}^{n} \mathbf{e}_i \frac{\partial}{\partial x_i}$. This operator is a hypercomplex analogue to the complex Cauchy-Riemann operator. In particular we have that $D^2 = -\Delta$, where Δ is the Laplacian over R^n. For a domain $\Omega \subset R^n$, a function $f : \Omega \mapsto C\ell_{0,n}$ is said to be *left-monogenic* if it satisfies the equation $Df = 0$. A similar definition can be given for right-monogenic functions. Basic properties of the Dirac operator and left-monogenic functions can be found in [1], [2], [6], and [5].

Now, we need some more facts for our discrete setting. To discretize point-wise the partial derivatives $\frac{\partial}{\partial x_i}$ in the equidistant lattice with mesh width $h > 0$, $R^n_h = \{mh = (m_1 h, \ldots, m_n h) : m \in Z^n\}$, we introduce forward/backward differences $\partial_h^{\pm i}$:

$$\partial_h^{\pm i} u(mh) = \mp \frac{u(mh) - u(mh \pm h\mathbf{e}_i)}{h} \tag{1}$$

These forward/backward differences $\partial_h^{\pm i}$ satisfy the following product rule

$$\begin{aligned} (\partial_h^{\pm i} fg)(mh) &= f(mh)(\partial_h^{\pm i} g)(mh) + (\partial_h^{\pm i} f)(mh) g(mh \pm h\mathbf{e}_i), && (2)\\ (\partial_h^{\pm i} fg)(mh) &= f(mh \pm h\mathbf{e}_i)(\partial_h^{\pm i} g)(mh) + (\partial_h^{\pm i} f)(mh) g(mh). && (3) \end{aligned}$$

The forward/backward discretizations of the Dirac operator are given by

$$D_h^{\pm} = \sum_{i=1}^{n} \mathbf{e}_i \partial_h^{\pm i}. \tag{4}$$

In the following we will also use the following multi-index abbreviations:

$$\begin{aligned} (mh)^{(\alpha)} &:= (m_1 h)^{\alpha_1} (m_2 h)^{\alpha_2} \ldots (m_n h)^{\alpha_n};\\ \alpha! &:= \alpha_1! \alpha_2! \ldots \alpha_n!;\\ |\alpha| &:= \alpha_1 + \alpha_2 + \ldots \alpha_n\\ \partial_h^{\pm \mathbf{e}_i} &:= \partial_h^{\pm i};\\ \partial_h^{\pm \alpha_i \mathbf{e}_i} &:= \left(\partial_h^{\pm \mathbf{e}_i}\right)^{\alpha_i}, \end{aligned}$$

for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{i=1}^{n} \mathbf{e}_i \alpha_i$.

3 FISCHER DECOMPOSITION

The basic idea of a Fischer decomposition is to decompose any homogeneous polynomial into monogenic homogeneous polynomials of lower degrees. In the classic case such a decomposition is based on the fact that the powers x^s are homogeneous and that $\frac{\partial x_i^s}{\partial x_i} = s x_i^{s-1}$. A first idea would be to consider instead of x^s simply the powers $(mh)^s$, but while these powers are still homogeneous the last condition is not true in the discrete case, unfortunately. Therefore, we will start by introducing discrete homogeneous powers which will play the equivalent role of x^s in the discrete case.

3.1 Multi-index factorial powers

Starting from the one-dimensional factorial powers

$$(m_i h)_{\mp}^{(0)} := 1, (m_i h)_{\mp}^{(s)} := \prod_{k=0}^{s-1}(m_i h \mp kh), s \in N \tag{5}$$

we introduce the multi-index factorial powers of degree $|\alpha|$ by

$$(mh)_{\mp}^{(\alpha)} = \prod_{i=1}^{n}(m_i h)_{\mp}^{(\alpha_i)}.$$

The one-dimensional factorial powers $(m_i h)_{\mp}^{(s)}$ have the following properties

P1. $(m_i h)_{\mp}^{(s+1)} = (m_i h \mp sh)(m_i h)_{\mp}^{(s)}$;

P2. $\partial_h^{\pm j}(m_i h)_{\mp}^{(s)} = s(m_i h)_{\mp}^{(s-1)}\delta_{i,j}$;

P3. $\partial_h^{\mp j}(m_i h)_{\mp}^{(s)} = s(m_i h \mp h)_{\mp}^{(s-1)}\delta_{i,j}$;

P4. $(m_i h)_{\mp}^{(s)} \to x_i^s = (m_i h)^s$ for $h \to 0$,

where $\delta_{i,j}$ denotes the standard Kronecker symbol.

As a direct consequence of these properties, we obtain the following lemmas:

Lemma 3.1 *The multi-index factorial powers of degree $|\alpha|$, $(mh)_{\mp}^{(\alpha)}$, satisfy*

$$\sum_{i=1}^{n}(m_i h)\partial_h^{\pm i}(mh \mp h\mathbf{e}_i)_{\mp}^{(\alpha)} = |\alpha|(mh)_{\mp}^{(\alpha)}$$

Lemma 3.2 *The multi-index factorial powers of degree $|\alpha|$, $(mh)_{\mp}^{(\alpha)}$, satisfy*

$$\partial_h^{\pm\beta}(mh)_{\mp}^{(\alpha)} = \alpha!\delta_{\alpha,\beta}.$$

Lemma 3.3 *The multi-index factorial powers of degree $|\alpha|$ approximate the classical multi-index powers of degree $|\alpha|$, that is*

$$(mh)_{\mp}^{(\alpha)} \to x^{(\alpha)} = (mh)^{(\alpha)} \; for \; h \to 0.$$

For all what follows, let $\Pi_d^{\pm}$ denote the space of all Clifford-valued polynomials of degree d, $P_d^{\pm}$, generated by the powers $(mh)_{\mp}^{(\alpha)}$ of degree $|\alpha| = d$, and $\Pi^{\pm}$ be the countable union of all Clifford-valued polynomials of degree $d \geq 0$. Furthermore, let $\mathcal{M}_d^{\pm} = \Pi_d^{\pm} \cap \ker D_h^{\pm}$ be the space of discrete monogenic polynomials of degree d. Based on Lemma 3.1, 3.2 and 3.3, we will show that it is possible to obtain discrete versions for the Fisher decomposition as well as define discrete versions of the Euler and Gamma operators.

3.2 The main theorem

For two Clifford-valued polynomials of degree d, $P_d^\pm$ and $Q_d^\pm \in \Pi_d^\pm$ given by

$$\begin{aligned} P_d^\pm(mh) &= \sum_{|\alpha|=d} (mh)_\mp^{(\alpha)} a_\alpha^\pm \\ Q_d^\pm(mh) &= \sum_{|\alpha|=d} (mh)_\mp^{(\alpha)} b_\alpha^\pm \end{aligned}$$

we define the Fischer inner product by

$$[P_d^\pm, Q_d^\pm]_h := \sum_{|\alpha|=d} \alpha! \mathrm{Sc}\,(\overline{a_\alpha^\pm} b_\alpha^\pm). \tag{6}$$

Denoting $P_d^\pm(D_h^\pm)$ the difference operator obtained from $P_d^\pm$ by replacing $m_i h$ by $\partial_h^{\pm i}$ (c.f. [2]), we have by Lemma 3.2 the identity

$$[P_d^\pm, Q_d^\pm]_h := \mathrm{Sc}\,(\overline{P_d^\pm(D_h^\pm)} Q_d^\pm)(0) \qquad P_d^\pm, Q_d^\pm \in \Pi_d^\pm. \tag{7}$$

Moreover, due to $\overline{D_h^\pm} = -D_h^\pm$ the Fischer inner product has the important property:

$$[(mh)P_d^\pm, Q_d^\pm]_h = -[P_d^\pm, D_h^\pm Q_d^\pm]_h. \tag{8}$$

This property combined with the inclusion property

$$D_h^\pm \Pi_d^\pm := \{D_h^\pm P_d^\pm : P_d^\pm \in \Pi_d^\pm\} \subset \Pi_{d-1}^\pm. \tag{9}$$

allows us to prove the following theorem:

Theorem 3.1 *We have*

$$\Pi_d^\pm = \mathcal{M}_d^\pm + (mh)\Pi_{d-1}^\pm.$$

Moreover, the subspaces $\mathcal{M}_d^\pm$ and $(mh)\Pi_{d-1}^\pm$ are orthogonal with respect to the Fischer inner product.

From this theorem we obtain the Fischer decomposition with respect to our difference Dirac operators $D_h^\pm$.

Theorem 3.2 Fischer decomposition *Let $P_d^\pm \in \Pi_d^\pm$ then*

$$P_d^\pm(mh) = \sum_{s=0}^{d-1} (mh)^s M_{d-s}^\pm(mh). \tag{10}$$

where each $M_j^\pm$ denotes the homogeneous discrete monogenic polynomials of degree j with respect to the Dirac operators $D_h^\pm$.

3.3 Difference Euler and Gamma operators

Based on Lemma 3.1 we will introduce discrete versions of the Euler and Gamma operators presented in [2].

First of all, we introduce the second order difference operator $A_h^{\pm}$ by

$$A_h^{\pm} = \mp h \sum_{i=1}^{n} (m_i h) \partial_h^{\pm i} \partial_h^{\mp i}. \tag{11}$$

Definition 3.1 *For a lattice function $f_h : R_h^n \to C\ell_{0,n}$, we introduce the difference Euler operator $E_h^{\pm}$ by*

$$(E_h^{\pm} f_h)(mh) = \sum_{i=1}^{n} (m_i h)(\partial_h^{\pm i} f_h)(mh \mp h\mathbf{e}_i)$$

and the difference Gamma operator $\Gamma_h^{\pm}$ by

$$(\Gamma_h^{\pm} f_h)(mh) = -\sum_{j<k} \mathbf{e}_j \mathbf{e}_k (L_{jk}^{\pm} f_h)(mh) - (A_h^{\pm} f_h)(mh),$$

where $L_{jk}^{\pm} := (m_j h)\partial_h^{\pm k} - (m_k h)\partial_h^{\pm j}$.

It looks surprising that we have in the definition of the Gamma operator a term which contains second order differences, but we would like to remark that for $h \to 0$ this term vanishes and we will get the usual continuous Gamma operator. As a matter of fact this term arises due to the fact that in the discrete case translations are involved in the definition of finite differences/ finite difference operators.

Using the definition of the difference Euler operator and Lemma 3.1, we obtain for polynomials homogeneous of degree d, $P_d^{\pm} \in \Pi_d^{\pm}$ $E_h^{\pm} P_d^{\pm} = dP_d^{\pm}$, and, moreover, we can show that a function f_h homogeneous of degree d satisfy $E_h^{\pm} f_h = df_h$. This fact provides a good motivation for calling $E_h^{\pm}$ Euler operator, i.e. an operator who measures the degree of homogeneity of a homogeneous function.

It follows from the definition of the Euler and Gamma operator that

$$\begin{aligned} (mh) D_h^{\pm} f_h &= -\sum_{i=1}^{n} (m_i h) \partial_h^{\pm i} f_h + \sum_{j<k} \mathbf{e}_j \mathbf{e}_k L_{jk}^{\pm} f_h & (12) \\ &= -(E_h^{\pm} + \Gamma_h^{\pm}) f_h & (13) \end{aligned}$$

Moreover, for discrete monogenic polynomials of degree d, $M_d^{\pm} \in \mathcal{M}_d^{\pm}$, we have $\Gamma_h^{\pm} M_d^{\pm} = -dM_d^{\pm}$.

For all what follows, we introduce the difference operators

$$B_h^{\pm} = \pm h \sum_{i=1}^{n} \partial_h^{\pm i}, \tag{14}$$

$$R_{h,r}^{\pm} = rI + E_h^{\pm} - A_h^{\pm}, \tag{15}$$

$$V_{h,r}^{\pm} = R_{h,r}^{\pm} + \frac{1}{2} B_h^{\pm}. \tag{16}$$

where I is the identity operator and r a real number.

From the identity

$$\begin{aligned}\left((mh)D_h^{\pm} + D_h^{\pm}(mh)\right) f_h &= -2(E_h^{\pm} - A_h^{\pm})f_h - nf_h \\ &= -2R_{h,n/2}^{\pm} f_h\end{aligned}$$

we get

$$\left(D_h^{\pm}(mh)\right) f_h = (-2R_{h,n/2}^{\pm} + E_h^{\pm} + \Gamma_h^{\pm})f_h, \tag{17}$$

by applying identity (13).

Applying the product rule for finite differences (2) and using the identity $-2m_i h = \mathbf{e}_i(mh) + (mh)\mathbf{e}_i$, $i = 1, \ldots, n$, we get the following propositions

Proposition 3.1 *For a lattice function $f_h : R_h^n \to C\ell_{0,n}$, we have*

$$D_h^{\pm} E_h^{\pm} f_h = D_h^{\pm} f_h + E_h^{\pm} D_h^{\pm} f_h.$$

Proposition 3.2 *For a lattice function $f_h : R_h^n \to C\ell_{0,n}$, we have*

$$D_h^{\pm}((mh)f_h) = -2V_{h,n/2}^{\pm} f_h - (mh)D_h^{\pm} f_h \tag{18}$$

The details of the proofs can be found in [4].

From proposition 3.2 and from the commutation properties $D_h^{\pm} A_h^{\pm} = A_h^{\pm} D_h^{\pm}$ and $D_h^{\pm} B_h^{\pm} = B_h^{\pm} D_h^{\pm}$ follow the operator relations

$$D_h^{\pm} R_{h,r}^{\pm} = R_{h,r+1}^{\pm} D_h^{\pm}, \tag{19}$$

$$D_h^{\pm} V_{h,r}^{\pm} = V_{h,r+1}^{\pm} D_h^{\pm}. \tag{20}$$

Combining proposition 3.2 with the operator relation (20), we obtain by recursion [4], the formula

$$(D_h^{\pm})^s((mh)^s M_d^{\pm}) \;=\; (-2)^s V_{h,n/2+s-2}^{\pm} V_{h,n/2+s-3}^{\pm} \ldots V_{h,n/2}^{\pm} M_d^{\pm}. \tag{21}$$

where $M_d^{\pm} \in \mathcal{M}_d^{\pm}$. From this follows also $(mh)^s M_d^{\pm} \in \ker(D_h^{\pm})^{s+1}$. Formula (21) gives us a motivation to find explicit formulae for the polynomials $M_d^{\pm}$. To this end we need an explicit formula for the inverse of the iterated composite operator $V_{h,n/2+s-2}^{\pm} V_{h,n/2+s-3}^{\pm} \ldots V_{h,n/2}^{\pm}$. This means that we have to find an explicit formula for the inverse of the operator $V_{h,r}^{\pm}$. Unfortunately, we are only able to get an explicit formula for the operator $R_{h,r}^{\pm}$. ([4]).

Theorem 3.3 *For a lattice function $f_h : R_h^n \to C\ell_{0,n}$ and for $r > 0$, the difference operator $J_{h,r}^{\pm}$ defined by*

$$(J_{h,r}^{\pm} f_h)(mh) = \sum_{th \in [0,1]_h^{\pm}} h d_h^{\pm} \left((th \mp h)_{\mp}^{(r-1)} f_h((th)(mh))\right)$$

satisfies

$$J_{h,r}^{\pm} R_{h,r}^{\pm} = I = R_{h,r}^{\pm} J_{h,r}^{\pm}.$$

Hereby we denote $[0,1]_h^{+} = [0,1)_h$, $[0,1]_h^{-} = (0,1]_h$, and

$$(d_h^{\pm} g)(th) := \mp \frac{g(th) - g(th \pm h)}{h}.$$

The main idea of the proof is based on the identity

$$f_h(mh) = \sum_{th \in [0,1]_h^{\pm}} h d_h^{\pm} \left((th \mp h)_{\mp}^{(r)} f_h((th)(mh))\right).$$

and on the application of the discrete version of the chain rule.

4 A DISCRETE HARMONIC FISCHER DECOMPOSITION

According to the classical theory of the finite differences, the usual approximation of the Laplacian is given by

$$\begin{aligned}(\Delta_h u)(mh) &= \sum_{i=1}^{n} \frac{u(mh+h\mathbf{e}_i)+u(mh-h\mathbf{e}_i)-2u(mh)}{h^2} \\ &= \sum_{i=1}^{n} (\partial_h^{\mp i}\partial_h^{\pm i} u)(mh). \end{aligned} \tag{22}$$

The first problem that arises now is that not all of our partial difference operators do commute in the certain sense (c.f. [5, 6]) and, moreover, we have no factorization of the discrete Laplacian Δ_h by means of our difference Dirac operators considered above, that is $D_h^{\mp} D_h^{\pm} \neq -\mathbf{e}_0\Delta_h$.

Let us restrict ourselves in this section to the case of quaternion-valued functions defined on lattices in R^3.

Let us remark that the quaternionic variable mh is identified with the 4×4 matrix

$$mh = \begin{pmatrix} 0 & -m_1h & -m_2h & -m_3h \\ m_1h & 0 & -m_3h & m_2h \\ m_2h & m_3h & 0 & m_1h \\ m_3h & -m_2h & m_1h & 0 \end{pmatrix}.$$

In [7] for a lattice function $f_h : R_h^3 \to H$ given by

$$f_h = \sum_{i=0}^{3} f_h^i \mathbf{e}_i = f_h^0\mathbf{e}_0 + \mathrm{Vec}\, f_h$$

a finite difference approximation of our Dirac operator was defined in the form

$$\begin{aligned} D_h^{-+} f_h &= \begin{pmatrix} 0 & -\partial_h^{-1} & -\partial_h^{-2} & -\partial_h^{-3} \\ \partial_h^{-1} & 0 & -\partial_h^{3} & \partial_h^{2} \\ \partial_h^{-2} & \partial_h^{3} & 0 & -\partial_h^{1} \\ \partial_h^{-3} & -\partial_h^{2} & \partial_h^{1} & 0 \end{pmatrix} \begin{pmatrix} f_h^0 \\ f_h^1 \\ f_h^2 \\ f_h^3 \end{pmatrix} \\ &= \begin{pmatrix} -\mathrm{div}_h^- \mathrm{Vec}\, f_h \\ \mathrm{grad}_h^- f_h^0 + \mathrm{curl}_h^+ \mathrm{Vec}\, f_h \end{pmatrix} \end{aligned} \tag{23}$$

$$\begin{aligned} D_h^{+-} f_h &= \begin{pmatrix} 0 & -\partial_h^{1} & -\partial_h^{2} & -\partial_h^{3} \\ \partial_h^{1} & 0 & -\partial_h^{-3} & \partial_h^{-2} \\ \partial_h^{2} & \partial_h^{-3} & 0 & -\partial_h^{-1} \\ \partial_h^{3} & -\partial_h^{-2} & \partial_h^{-1} & 0 \end{pmatrix} \begin{pmatrix} f_h^0 \\ f_h^1 \\ f_h^2 \\ f_h^3 \end{pmatrix} \\ &= \begin{pmatrix} -\mathrm{div}_h^+ \mathrm{Vec}\, f_h \\ \mathrm{grad}_h^+ f_h^0 + \mathrm{curl}_h^- \mathrm{Vec}\, f_h \end{pmatrix} \end{aligned} \tag{24}$$

with $\mathrm{div}_h^{\pm}\mathrm{Vec}\, f_h = \sum_{i=1}^3 \partial_h^{\pm i} f_h^i$, $\mathrm{grad}_h^{\pm} f_h^0 = \sum_{i=1}^3 (\partial_h^{\pm i} f_h^0)\mathbf{e}_i$ and

$$\mathrm{curl}_h^{\pm}\mathrm{Vec}\, f_h = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial_h^{\pm 1} & \partial_h^{\pm 2} & \partial_h^{\pm 3} \\ f_h^1 & f_h^2 & f_h^3 \end{vmatrix}.$$

In the latter form one can easily see the similarity with the usual Dirac operator

$$Df = \begin{pmatrix} -\mathrm{div}\mathrm{Vec}\, f \\ \mathrm{grad}\mathrm{Sc}\, f + \mathrm{curl}\mathrm{Vec}\, f \end{pmatrix}.$$

and we obtain the following factorization of the discrete Laplacian

$$D_h^{+-} D_h^{-+} f_h = \begin{pmatrix} -\Delta_h f_h^0 \\ -\Delta_h \mathrm{Vec}\, f_h \end{pmatrix} = D_h^{-+} D_h^{+-} f_h. \tag{25}$$

Now, we are able to obtain a Fischer decomposition for the discrete Dirac operators D_h^{-+} and D_h^{+-}.

Proving the inclusion properties $D_h^{+-}\Pi_d^+ \subset \Pi_{d-1}^+$, $D_h^{-+}\Pi_d^- \subset \Pi_{d-1}^-$ and replacing D_h^{+-} by D_h^+ and D_h^{-+} by D_h^- in the inner product (7), we obtain the Fischer decompositions:

Theorem 4.1 Fischer decomposition for D_h^{-+} and D_h^{+-}

Let $P_d^- \in \Pi_d^-$ (respectively, $P_d^+ \in \Pi_d^+$) then

$$P_d^- = \sum_{s=0}^{d-1} (mh)^s M_{d-s}^{-+}, \tag{26}$$

$$P_d^+ = \sum_{s=0}^{d-1} (mh)^s M_{d-s}^{+-}. \tag{27}$$

where each M_j^{-+}(respectively, M_j^{+-}) denotes a homogeneous discrete monogenic polynomial of degree j, that is, $M_j^{-+} \in \Pi_j^- \cap \ker D_h^{-+}$(respectively, $M_j^{+-} \in \Pi_j^+ \cap \ker D_h^{+-}$).

From the factorization property (25), we have

$$[(mh)^2 P_d^{\pm}, Q_d^{\pm}]_h = -[P_d^{\pm}, \Delta_h Q_d^{\pm}]_h,$$

which allows us to obtain the Fischer decomposition for the discrete Laplacian:

Theorem 4.2 Fischer decomposition for Δ_h

Let $P_d^{\pm} \in \Pi_d^{\pm}$ then

$$P_d^{\pm} = \sum_{2s \leq d} |mh|^{2s} \mathcal{H}_{d-2s}^{\pm},$$

where each $\mathcal{H}_j^{\pm}$ denotes a homogeneous discrete harmonic polynomial of degree j, that is, $\mathcal{H}_j^{\pm} \in \Pi_d^{\pm} \cap \ker \Delta_h$.

As a consequence of the theorem 4.1, we obtain Fischer decompositions which relate the discrete harmonic and the discrete monogenic polynomials.

Corollary 4.1 Fischer decomposition *Let $\mathcal{H}_d^{\pm} \in \Pi_d^{\pm} \cap \ker \Delta_h$ then*

$$\mathcal{H}_d^- = M_d^{-+} + (mh) M_{d-1}^{-+}, \tag{28}$$

$$\mathcal{H}_d^+ = M_d^{+-} + (mh) M_{d-1}^{+-}. \tag{29}$$

where each M_j^{-+}(respectively, M_j^{+-}) denotes a homogeneous discrete monogenic polynomial of degree j, that is, $M_j^{-+} \in \Pi_j^- \cap \ker D_h^{-+}$(respectively, $M_j^{+-} \in \Pi_j^+ \cap \ker D_h^{+-}$).

Using the same ideas as in the subsection 3.3, we can define the Euler and Gamma operators E_h^{+-}, Γ_h^{+-} (respectively, E_h^{-+}, Γ_h^{-+}) for the modified Dirac operators D_h^{+-} (respectively, D_h^{-+}) which satisfy the identity $(mh)D_h^{-+} = -E_h^{-+} - \Gamma_h^{-+}$ (respectively, $(mh)D_h^{+-} = -E_h^{+-} - \Gamma_h^{+-}$) [4]. Moreover, the polynomials $P_k^{\pm} \in \Pi_k^{\pm}$ satisfy $E_h^{-+}P_k^- = kP_k^-$, (respectively, $E_h^{+-}P_k^+ = kP_k^+$) and when $P_k^- \in \ker D_h^{-+}$ (respectively, $P_k^+ \in \ker D_h^{+-}$), we obtain $\Gamma_h^{-+}P_k^- = -kP_k^-$, (respectively, $\Gamma_h^{+-}P_k^+ = -kP_k^+$).

Like in the proposition 3.2 we can prove the operator property $D_h^{-+}E_h^{-+} = I + E_h^{-+}D_h^{-+}$ (respectively, $D_h^{+-}E_h^{+-} = I + E_h^{+-}D_h^{+-}$). In the same way we get analogous relations to the ones presented in subsection 3.3.

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